Basis for the Proposition of the $I\Delta$ and Δ Functions of the STP Method.

Technical Report.

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1 Basis for the Proposition of the *I*Δ and Δ Functions of the STP Method.

In the STP method the output rate of each queue maintains proportionality to its average relative length. To clarify this idea, the following paragraphs compare two queuing settings analyzing their outcomes.

1.1 Proportionality of the Average Relative Length.

Case 1.

This first case considers a node that receives traffic from two routes arriving, respectively, through the input interfaces: *A* and *B*. As depicted in Figure 1, the traffic from these routes leaves the node through a single output interface.

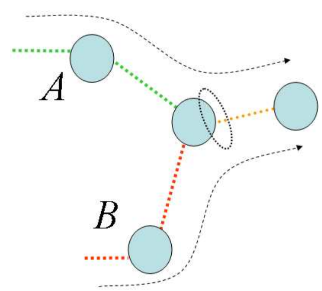

Figure 1. Representation of two routes which arrive to a node through its input interfaces: *A* and *B*, and leave the node through its single output interface.

The rates of the incoming traffic on each route are r_A and r_B, respectively, and are kept constant. The output interface has a single FIFO queue, and its output rate is R, when busy[1].

The traffic coming from the two routes would have an average long-term relative contribution to the total length of the queue of: $r_A / (r_A + r_B)$, and $r_B / (r_A + r_B)$, respectively (for the proof of this statement –under certain assumptions –see *Theorem 1* in Appendix 1). In the long-term, if the average total length of the traffic waiting to be served at the queue is Q, then the average contribution to that length, on behalf of each one of the two routes, is: $Q_A = Q\, r_A / (r_A + r_B)$ and $Q_B = Q\, r_B / (r_A + r_B)$, respectively, where $Q = Q_A + Q_B$.

If R_A and R_B represent the average output rates of the traffic that has come from interfaces *A* and *B*, respectively, then the mean time to serve the contents of the queue is: $Q / R = Q_A / R_A = Q_B / R_B$. It follows that

$$R_A = R\,Q_A / (Q_A + Q_B) = R\,r_A / (r_A + r_B)$$
$$R_B = R\,Q_B / (Q_A + Q_B) = R\,r_B / (r_A + r_B) \quad (1)$$

[1] All the queues considered in this paper are work-conserving: they work at full rate whenever there is data to be sent.

Therefore, the relative output rate of each source is proportional to the relative contribution of its corresponding input rate.

Case 2.

This second case is taken from the previous one but now the output-interface of the node in Figure 1 has two queues, *bA* and *bB*, that hold, respectively, the traffic arriving from input interfaces *A* and *B*. The average lengths of these queues are: Q_A and Q_B, respectively.

The scheduler of the output-interface follows the WFQ algorithm with the particular characteristic that the weights of the queues are proportional to their relative average lengths. For the two-queue example, the corresponding weights would be:

$$\begin{aligned} \phi_A &= Q_A / (Q_A + Q_B) \\ \phi_B &= Q_B / (Q_A + Q_B) \end{aligned} \qquad \mathbf{2}$$

Therefore, the respective output rates would be:

$$\begin{aligned} R_A &= R\phi_A = RQ_A / (Q_A + Q_B) \\ R_B &= R\phi_B = RQ_B / (Q_A + Q_B) \end{aligned} \qquad \mathbf{3}$$

Comparing equations 1 and 3 it can be deduced that the relative output rates of the traffic accumulated in the single queue system are equal to those of the multiple queue system where the individual output rates are proportional to their relative queue size.

Figure 2 illustrates this burst arriving to queue *bA*, at a time when the lengths of the queues are q_A and q_B, respectively. From equation 3 it can be seen that, in the average, a small burst of traffic would take a time: $t_A = Q_A / R_A = t_B = Q_B / R_B = Q / R$, from its arrival until it is completely served.

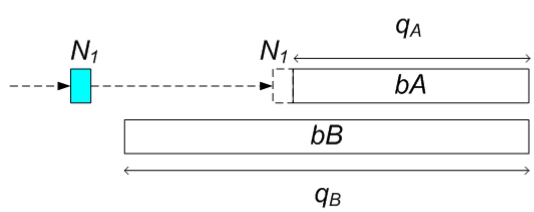

Figure 2. Representation of the instantaneous lengths, q_A and q_B, of two queues: *bA* and *bB*, in an output interface at the time where a burst N_1 of traffic arrives to queue *bA*.

Conclusions.

The time to serve an arriving burst of traffic is the same for both queues when the weight of each queue is proportional to its relative length.

This condition can be taken to be a fair attention policy for the queues.

For example, from equation 3, the weight of queue *A* is ϕ_A, and its relative average length is $Q_A / (Q_A + Q_B)$.

With the above considerations, in a multiple-queue output interface, where the relative average length of queue *i*, at time $t_0+(r+1)\tau$ is: $Q_i\left(t_0+(r+1)\tau\right)\Big/\sum_{j=1}^{N}Q_j\left(t_0+(r+1)\tau\right)$, and the weight it has had, within the time interval $t_0+r\tau$ to $t_0+(r+1)\tau$, is $\phi_i^{Act}\left(t_0+r\tau\right)$, an indicator to know whether this weight should increase of not is proposed as:

$$I\Delta\left(\phi_i^{Act}\left(t_0+r\tau\right)\right)=\frac{Q_i\left(t_0+(r+1)\tau\right)}{\sum_{j=1}^{N}Q_j\left(t_0+(r+1)\tau\right)}-\phi_i^{Act}\left(t_0+r\tau\right)$$

More insight into the meaning of this proposed indicator, *IΔ* is given in Appendix 2 and Appendix 3.

More insight into the development of the *Δ* function for the STP method is given in Appendix 4.

Appendix 1. The Relation of Traffic-Contribution.

Consider a case of fluid traffic with a single FIFO, work-conserving queue, big enough so as to a-void overflows. The queue is fed by two sources A and B, with constant rates r_A and r_B, respectively. Consider that the queue is stable[2]. Consider that the time resets to 0 at the beginning of any busy period of the queue.

At time t, the amount of traffic that has left the queue is $\gamma(t) = \gamma_A(t) + \gamma_B(t)$, where the sub index indicates if the data has come from source A or B.

At time $t^+ > 0$ the total amount of fluid data at the queue is $Q(t^+) = Q_A(t^+) + Q_B(t^+)$, where Q_A and Q_B are the fluid data amounts corresponding to sources A and B, respectively.

It is clear then that $Q_A(t^+) = r_A t^+ - \gamma_A(t)$ and $Q_B(t^+) = r_B t^+ - \gamma_B(t)$; and also that $= r_A t^+ + r_B t^+ = Q(t^+) + \gamma(t)$.

Consider that the fluid data is well behaved, such that the amount that has arrived to the queue has the same value at t and at t^+. As $t >> 0$, $\gamma_A(t)$ and $\gamma_B(t)$ become very big.

Theorem 1. The Relation of Traffic-Contribution.

For $t >> 0$, the mean value of $Q_A(t) / Q(t)$ tends to be equal to $r_A / (r_A + r_B)$, and the mean value of $Q_B(t) / Q(t)$ tends to be equal to $r_B / (r_A + r_B)$.

Proof:

At $t >> 0$, $\gamma_A(t) / \gamma(t) = (r_A t - Q_A(t)) / ((r_A + r_B) t - Q(t)) \approx r_A / (r_A + r_B)$, and $\gamma_B(t) / \gamma(t) \approx r_B / (r_A + r_B)$, as $r_A t >> Q_A(t)$, and $r_B t >> Q_B(t)$, so that that the random variation of $\gamma_A(t) / \gamma(t)$ dwindles as t increases. Then, for $t >> 0$, the mean values of $\gamma_A(t)$ and $\gamma_B(t)$ would be $\overline{\gamma_A(t)} = \overline{\gamma(t)}\, r_A / (r_A + r_B)$ and $\overline{\gamma_B(t)} = \overline{\gamma(t)}\, r_B / (r_A + r_B)$, respectively.

With the above approximation, for $t >> 0$, it is obtained that:

$Q_A(t) = r_A t - \gamma_A(t) = r_A t - \gamma_A(t)\left[\overline{\gamma(t)}/\overline{\gamma(t)}\right] \approx r_A t - \gamma(t)(r_A / (r_A + r_B))$.

This last result is defined as $\tilde{Q}_A(t)$ which is a random variable with a mean value which gets close to that of $Q_A(t)$ as t grows very large. To see why, the difference $Q_A(t) - \tilde{Q}_A(t)$ is formed. Developing this difference (valid for $t >> 1$): $Q_A(t) - \tilde{Q}_A(t) = r_A t - \gamma_A(t) - r_A t + \gamma(t)(r_A / (r_A + r_B))$ $= = - \gamma_A(t) + (\gamma_A(t) + \gamma_B(t))(r_A / (r_A + r_B)) = \gamma_A(t))(r_A / (r_A + r_B) - 1) + \gamma_B(t))(r_A / (r_A + r_B)) =$ $\gamma_B(t)(r_A / (r_A + r_B)) - \gamma_A(t)(r_B / (r_A + r_B))$. Obtaining the mean value of this difference results in

[2] That is, the total input rate and the output departure process are such that the queue empties infinitely often, with probability one (see M. Neely, Instructor. Lecture Notes. EE 549, from lecture 1 to 5, University of Southern California, (2006), Available: http://www-rcf.usc.edu/~mjneely/ee549notes/EE549_Supplementary_Lecture_Notes_01.pdf, Jun. 2010).

$\overline{\gamma_B(t)}\,(r_A / (r_A + r_B)) - \overline{\gamma_A(t)}\,(r_B / (r_A + r_B))$, and substituting theses mean values with the values obtained above, for $\gamma_A(t)$ and $\gamma_B(t)$, for t >>0, gives a result of "zero". Something similar would happen for the difference $Q_B(t) - \tilde{Q}_B(t)$. And forming $\tilde{Q}(t) = \tilde{Q}_A(t) + \tilde{Q}_B(t)$, it can be shown that the mean value of $\tilde{Q}(t)$ gets close to that of $Q(t)$ as t grows very large.

Now, developing the expression $\tilde{Q}_A(t)$:

$\tilde{Q}_A(t) = r_A t\left(1-\frac{\gamma(t)}{t(r_A+r_B)}\right) = r_A t\left(1-\frac{\gamma(t)}{Q(t)+\gamma(t)}\right) = r_A t\left(\frac{Q(t)}{Q(t)+\gamma(t)}\right) \approx r_A\, t\, Q(t) / \gamma(t)$, as $\gamma(t)$ >> $Q(t)$ for t >> 1, so, for t >> 0, $\tilde{Q}_A(t) / Q(t) \approx r_A\, t / \gamma(t)$. In the same form it is obtained that $\tilde{Q}_B(t) / Q(t) \approx r_B\, t / \gamma(t)$. Mixing these equations: $\tilde{Q}_A(t) / (\tilde{Q}_A(t) + \tilde{Q}_B(t)) = \tilde{Q}_A(t) / \tilde{Q}(t) = r_A / (r_A + r_B)$. Similarly $\tilde{Q}_B(t) / (\tilde{Q}_A(t) + \tilde{Q}_B(t)) = \tilde{Q}_B(t) / \tilde{Q}(t) = r_B / (r_A + r_B)$.

With this, the mean values of $Q_A(t) / Q(t)$ and $Q_B(t) / Q(t)$ would also be equal to $r_A / (r_A + r_B)$ and $r_B / (r_A + r_B)$, respectively, as t becomes very large.

□

Appendix 2. The indicator IΔ in the STP Method. Its Deduction.

The indicator $I\Delta\left(\phi_i^{Act}\left(t_0 + r\,\tau\right)\right)$ of the STP method is the following.

$$I\Delta\left(\phi_i^{Act}\left(t_0 + r\,\tau\right)\right) = \frac{Q_i\left(t_0 + \left(r+1\right)\tau\right)}{\sum_{j=1}^{N} Q_j\left(t_0 + \left(r+1\right)\tau\right)} - \phi_i^{Act}\left(t_0 + r\,\tau\right) \qquad \mathbf{4}$$

The insight for proposing the equation for this indicator is the following.

Consider that the new weight of queue *i*, represented with $\phi_i^{New}\left(t_0 + \left(r+1\right)\tau\right)$, is calculated as a weighted average between the relative size of the current weight, at $t_0 + r\tau$, and the relative size of the average length of the queue at $t_0 + (r + 1)\,\tau$, as expressed in 5, where $0 < \alpha < 1$ and r is an integer such that $r \geq 0$.

$$\phi_i^{New}\left(t_0 + \left(r+1\right)\tau\right) = \left(1-\alpha\right)\frac{\phi_i^{Act}\left(t_0 + r\,\tau\right)}{\sum_{j=1}^{N}\phi_j^{Act}\left(t_0 + r\,\tau\right)} + \alpha\frac{Q_i\left(t_0 + \left(r+1\right)\tau\right)}{\sum_{j=1}^{N} Q_j\left(t_0 + \left(r+1\right)\tau\right)} \qquad \mathbf{5}$$

The smaller the parameter α the less sensible the new weights will be with regard to the relative average lengths of the queues. The divisor $\sum_{j=1}^{N}\phi_j^{Act}\left(t_0 + r\,\tau\right)$ should not be necessary as it has been supposed that its value is 1, but it is used anyway to assure that the new weights add up to 1 at every calculation.

If both members of equation 5 are summed, for $i = 1, \ldots, N$, the result is:

$$\sum_{i=1}^{N}\phi_i^{New}\left(t_0 + \left(r+1\right)\tau\right) = \left(1-\alpha\right)\frac{\sum_{i=1}^{N}\phi_i^{Act}\left(t_0 + r\,\tau\right)}{\sum_{j=1}^{N}\phi_j^{Act}\left(t_0 + r\,\tau\right)} + \alpha\,\frac{\sum_{i=1}^{N} Q_i\left(t_0 + \left(r+1\right)\tau\right)}{\sum_{j=1}^{N} Q_j\left(t_0 + \left(nr+1\right)\tau\right)} = 1.$$

So that the new weighs will add up to 1.

Equation 5 can be transformed considering that $\sum_{j=1}^{N}\phi_j^{Act}\left(t_0 + r\tau\right) = 1$ and expressing the substraction $\phi_i^{New}\left(t_0 + \left(r+1\right)\tau\right) - \phi_i^{Act}\left(t_0 + r\,\tau\right)$ as $\Delta\left(\phi_i^{Act}\left(t_0 + r\,\tau\right)\right)$. Then, equation 5 would look like the following:

$$\Delta\left(\phi_i^{Act}\left(t_0+r\tau\right)\right)=\alpha\left(\frac{Q_i\left(t_0+\left(r+1\right)\tau\right)}{\sum_{j=1}^{N}Q_m\left(t_0+\left(r+1\right)\tau\right)}-\phi_i^{Act}\left(t_0+r\tau\right)\right) \quad \textbf{6}$$

This equation is similar to equation 4. It is important to remember that equation 6 is not used to obtain a "Delta" function for the weight of queue *i*, but instead of that it is used to obtain an indicator to know if queue *i* should grow or decrease.

In the first experiments, previous to the proposition of the STP methods, an equation similar to equation 6 was used directly to obtain the "Delta" function for the weight of queue *i*. This had the problem that it was not be clear how to control the speed of convergence of the current weight $\phi_i^{Act}(\ )$ towards the objective value $Q_i(\ )\phi_i' / \sum_{j=1}^{N} Q_j(\ )\phi_j'$, as this convergence depended on the value of α, on the amount of updatings made per second, and on the difference between the current weight and the objective value, so another "Delta" function was proposed: that of the STP method.

Appendix 3. Initial Weights for the STP Method.

At t_0, the initial time, the weights of the queue are ϕ_i' for $i = 1, ..., N$, such that:

$$\phi_i^{Act}(t_0) = \left[\phi_i' \Big/ \sum_{j=1}^{N} \phi_j'\right],\ i = 1, \ldots, N .$$

These values are constant values called the "initial weights of the queues", which may be set depending on the relative importance given to a queue with regard to the others.
It is clear that $\sum_{j=1}^{N} \phi_j^{Act}(t_0) = 1$. With these weights, the indicator $I\Delta$ of equation 4 is modified as, such that the average lengths of the queues are always biased with the initial weights of the queues.

$$I\Delta\left(\phi_i^{Act}(t_0 + r\tau)\right) = \frac{Q_i\left(t_0 + (r+1)\tau\right)\phi_i'}{\sum_{j=1}^{N} Q_j\left(t_0 + (r+1)\tau\right)\phi_j'} - \phi_i^{Act}(t_0 + r\tau) \qquad \mathbf{7}$$

Appendix 4. The Deduction of the Δ Function of the STP Method.

Consider that a parameter "$(\tau / T) f$" is the percentage loss of weight of a queue i, supposing that it is designated by Δi of equation 4 as having to lose weight, in a step time of duration τ. For example, if $f = 0.2$ and $\tau / T = 1/100$ then this parameter should be equal to 0.002. Observe that this loss percentage is proportional to the length of the time duration considered.

To propose a form for Δ, following this criteria, in equation 4 the first term inside the parenthesis is changed by $\phi_i^{Act}\left(t_0 + r\,\tau\right)/\left(1-f\right)$, and the parameter α is changed by $\tau/T\left(1-f\right)$ (look that α is very small as it is supposed to be), obtaining:

$$\Delta\left(\phi_i^{Act}\left(t_0 + r\,\tau\right)\right) = \tau/T\left(1-f\right)\left(\phi_i^{Act}\left(t_0 + r\,\tau\right)/\left(1-f\right) - \phi_i^{Act}\left(t_0 + r\,\tau\right)\right) \qquad \textbf{8}$$

The equation 8 can be transformed in:

$\Delta\left(\phi_i^{Act}\left(t_0 + r\,\tau\right)\right) = \left(\tau/T\right) f\,\phi_i^{Act}\left(t_0 + r\,\tau\right)$, obtaining the expected result.

This result is written in the form used in the STP method, in equation 9.

$$\Delta\left(\phi_i^{Act}\left(t_0 + r\,\tau\right)\right) = -f\left(P\right)\frac{\tau}{T}\phi_i^{Act}\left(t_0 + r\,\tau\right) < 0 \qquad \textbf{9}$$

The fact that parameter f is a function of P is helpful and this fact is demonstrated in the development of the STP method itself.